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STRESS INTENSITY FACTORS FOR PARALLEL CRACKS LYING CLOSE TOGETHER IN A PLANE REGION*

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The asymptotic form of the stress intensity factors (SIF) at the tips of parallel cracks of different length is determined. The relative distance separating the cracks is regarded as a small parameter. The interaction between two or several cracks has been the subject of a large number of investigations (see the reviews in /1, 2/ etc.). Approximate formulas for the SIF are known for the case of cracks lying far apart from each other. The asymptotic formulas obtained in this paper relate to the opposite situation, in which the use of numerical methods in particular, meets with difficulties. Versions of the algorithms given in /3-5/ are used, and the results are expressed in terms of the solution of the problem of a single crack.

1. Formulation of the problem. Let Ω be a region in the plane $\mathbf{R}^2$ containing a segment $M = \{\mathbf{x} : x_2 = 0, |x_1| < a\}$. We will write $N_\varepsilon = \{\mathbf{x} : x_2 = \varepsilon, x_1 \in [-b_-, b_+]\}$ and use normalizing methods to reduce the characteristic size of the region Ω to unity. The Cartesian coordinates and the quantities ε, a, $b_\pm$ will then become dimensionless. We shall assume that the relative distance ε between the cracks is a small parameter of the problem, while the numbers a, $\pm b_\pm$, $a \mp b_\pm$ and the distance between M and $\partial\Omega$ are much greater than ε. We shall consider the problem of the plane deformation of a homogeneous isotropic body weakened by parallel cracks M and N_ε, situated close to each other. Let there be no mass forces, let the crack edges be stress-free, and let the body be acted upon by an external, selfbalanced load $\mathbf{p}$. The mathematical formulation of the problem is as follows:

$$L(\partial/\partial\mathbf{x})\,\mathbf{u}^\varepsilon(\mathbf{x}) \equiv \mu\Delta\mathbf{u}^\varepsilon(\mathbf{x}) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon = \Omega \setminus (M \cup N_\varepsilon) \tag{1.1}$$

$$\boldsymbol{\sigma}^{(n)}(\mathbf{u}^\varepsilon; \mathbf{x}) = \mathbf{p}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \tag{1.2}$$

$$\sigma_{12}(\mathbf{u}^\varepsilon; \mathbf{x}) = \sigma_{22}(\mathbf{u}^\varepsilon; \mathbf{x}) = 0, \quad \mathbf{x} \in M \cup N_\varepsilon \tag{1.3}$$

Here λ, μ are the Lamé coefficients, $\mathbf{u}^\varepsilon$ is the displacement vector, $\sigma(\mathbf{u}^\varepsilon)$ is the stress tensor, $\mathbf{n}$ is the unit vector of the outer normal, and $\boldsymbol{\sigma}^{(n)} = \sigma\cdot\mathbf{n}$.

Since $N_0 \subset M$, it follows that the region Ω_ε is transformed, as $\varepsilon \to 0$, into the region Ω_0 with a single crack M; $\Omega_0 = \Omega \setminus M$. Let us denote by $\mathbf{v}^\circ$ the solution of the corresponding problem

$$L(\partial/\partial\mathbf{x})\,\mathbf{v}^\circ(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_0; \quad \boldsymbol{\sigma}^{(n)}(\mathbf{v}^\circ; \mathbf{x}) = \mathbf{p}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \tag{1.4}$$

$$\sigma_{12}(\mathbf{v}^\circ; \mathbf{x}) = \sigma_{22}(\mathbf{v}^\circ; \mathbf{x}) = 0, \ \mathbf{x} \in M \tag{1.5}$$

Near the ends of the crack M the vector $\mathbf{v}^\circ$ can be represented in the form

$$\mathbf{v}^\circ(\mathbf{x}) = c^\pm + r_\pm^{1/2}(K_1^\pm\boldsymbol{\Phi}^1(\theta_\pm) + K_2^\pm\boldsymbol{\Phi}^2(\theta_\pm)) + O(r_\pm), \quad r_\pm \to 0 \tag{1.6}$$

$$(\Phi_r^1(\theta), \Phi_\theta^1(\theta)) = (4\mu)^{-1}(2\pi)^{-1/2}([2\varkappa - 1]\cos{}^1/_2\theta - \cos{}^3/_2\theta, \ \sin{}^3/_2\theta - [2\varkappa + 1]\sin{}^1/_2\theta)$$

$$(\Phi_r^2(\theta), \Phi_\theta^2(\theta)) = (4\mu)^{-1}(2\pi)^{-1/2}(3\sin{}^3/_2\theta - [2\varkappa - 1]\sin{}^1/_2\theta, \ 3\cos{}^3/_2\theta - [2\varkappa + 1]\cos{}^1/_2\theta)$$

$$\varkappa = (\lambda + 3\mu)(\lambda+\mu)^{-1}, \ \theta_\pm \in (-\pi, \pi) \tag{1.7}$$

Here $c^\pm$ is a constant vector, $(r_\pm, \theta_\pm)$ are polar coordinates with centre at the point $(\pm a, 0)$ and the polar axis is directed along M; $K_j^\pm$ is the SIF.

Representations analogous to (1.6) hold for the field $\mathbf{u}^\varepsilon$, and we denote the corresponding SIF by $K_j^\pm(\varepsilon)$. In addition, let $k_j^\pm(\varepsilon)$ be the SIF at the tip of the crack N_ε.

Below we shall find useful the representations of the field $\mathbf{v}^\circ$ near the points $(\pm b_\pm, 0)$,

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which should also be regarded as singular by virtue of the structure of the boundary of the initial region $\partial\Omega_\varepsilon$

$$\mathbf{v}^\circ(\mathbf{x}) = \sum_{j=1}^{6} l_{0j}^{+}\mathbf{U}^j(\mathbf{y}) + O(|\mathbf{y}|^3) \quad (\mathbf{y} = (x_1 - b_+, x_2)) \tag{1.8}$$

$$\begin{gathered}\mathbf{U}^k(\mathbf{y}) = \mathbf{e}^k, \quad l_{0k}^{+} = \mathbf{v}_k^{\circ}(b_+, +0) \quad (k = 1, 2)\\ \mathbf{U}^3(\mathbf{y}) = (-y_2, y_1), \quad \mathbf{U}^4(\mathbf{y}) = [4\mu(\lambda + \mu)]^{-1}((\lambda + 2\mu)y_1, -\lambda y_2)\\ l_{03}^{+} = {}^1/_2(\overset{\circ}{v}_{2,1}(b_+, +0) - \overset{\circ}{v}_{1,2}(b_+, +0)), \quad l_{04}^{+} = \sigma_{11}(\mathbf{v}^\circ; b_+, +0)\\ \mathbf{U}^5(\mathbf{y}) = [8\mu(\lambda + \mu)]^{-1}[\lambda + 2\mu](2y_1y_2, -y_1^2 - \lambda[\lambda + 2\mu]^{-1}y_2^2)\\ \mathbf{U}^6(\mathbf{y}) = [8\mu(\lambda + \mu)]^{-1}([\lambda + 2\mu]y_1^2 - [3\lambda + 4\mu]y_2^2, -2\lambda y_1y_2)\\ l_{05}^{+} = \sigma_{11,2}(\mathbf{v}^\circ; b_+, +0), \ l_{06}^{+} = \sigma_{11,1}(\mathbf{v}^\circ; b_+, +0)\end{gathered} \tag{1.9}$$

The index k following the comma denotes differentiation with respect to x_k. We have the same formulas for the point $(-b_-, 0)$ and $\mathbf{y} = (x_1 + b_-, x_2)$, while the coefficients of linear combination of the form (1.8) are denoted by l_{0j}^{-}.

The aim of this paper is to determine the asymptotic form of the solution $\mathbf{u}^\varepsilon$ of problem (1.1)-(1.3) as $\varepsilon \to 0$, and of the asymptotic form of the corresponding SIF.

2. ***The asymptotic form of the solution away from the crack*** N_ε. The vector $\mathbf{v}^\circ$ satisfies system (1.1) and boundary conditions (1.2), but leaves a discrepancy in conditions (1.3) at the edges N_ε^{+} of the crack N_ε. Expanding the stresses $\sigma_{j2}(\mathbf{v}^\circ)$ in a Maclaurin's series in the variables x_2, we can represent the above error in the form

$$\begin{gathered}\sigma_{j2}(\mathbf{v}^\circ; x_1, \varepsilon) = \varepsilon\sigma_{j2}(\mathbf{v}_{,2}^\circ; x_1, +0) + {}^1/_2\varepsilon^2\sigma_{j2}(\mathbf{v}_{,22}^\circ; x_1 + 0) + O(\varepsilon^3)\\ x_1 \in [-b_-, b_+], \ j = 1, 2\end{gathered} \tag{2.1}$$

Thus the principal term of the discrepancy (2.1) can be compensated with the help of the vector function $\varepsilon\mathbf{v}^1$, satisfying the equations

$$\begin{gathered}L(\partial/\partial\mathbf{x})\mathbf{v}^1(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_0; \quad \sigma^{(n)}(\mathbf{v}^1; \mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega\\ \sigma_{j2}(\mathbf{v}^1; \mathbf{x}) = q_j^{\pm}(\mathbf{x}), \quad \mathbf{x} \in M^{\pm}, \quad j = 1, 2\end{gathered} \tag{2.2}$$

The load $q^{\pm}$ is found from the relations

$$q^-(x_1) = 0, \ |x_1| < a; \ q^+(x_1) = 0, \ x_1 \in [-a, -b_-] \cup [b_+, a] \tag{2.3}$$

$$q_j^+(x_1) = -\sigma_{j2}(\mathbf{v}_{,2}^\circ; x_1, +0), \ x_1 \in [-b_-, b_+] \tag{2.4}$$

Taking relations (1.1) and (1.3) into account, we conclude that the principal vector and moment of the load (2.3), (2.4) are given by the formulas

$$\begin{gathered}T_1^1 = -\int_{-b_-}^{b_+} \frac{\partial\sigma_{12}}{\partial x_2}(\mathbf{v}^\circ; x_1, +0)\,dx_1 = \int_{-b_-}^{b_+} \frac{\partial\sigma_{11}}{\partial x_1}(\mathbf{v}^\circ; x_1, +0)\,dx_1 =\\ \sigma_{11}(\mathbf{v}^\circ; b_+, +0) - \sigma_{11}(\mathbf{v}^\circ; -b_-, +0); \quad T_2^1 = 0, \quad R^1 = 0\end{gathered} \tag{2.5}$$

According to (2.5), the load (2.3), (2.4) is in general non-selfbalanced. Therefore problem (2.2) has no solution possessing a finite elastic energy. If we extend the class of solutions to admit vectors with singularities of the order of $O(\ln|\mathbf{x} - \mathbf{P}^{\pm}|^{-1})$ at the ends $\mathbf{P}^{\pm} = (\pm b_{\pm}, 0)$ of the segment $N_{o\varepsilon}$ then problem (2.2) will become solvable and the solution will not be unique. Indeed, the class is extended on account of the addition, at the points $\mathbf{P}^{\pm}$, of unknown concentrated forces $\mathbf{T}^{\pm}$, which balance the action of the load (2.3), (2.4). However, a solution $\mathbf{V}^1$ exists of the problem of the deformation of a region Ω_0 by concentrated forces $\mathbf{T}^{\pm} = (\pm 1, 0)$ (their principal vector and principal moment are both equal to zero), i.e. the choice of the components $T_j^{\pm}$ is not unique. Thus we should take, as a solution of problem (2.2), the sum

$$\mathbf{v}^1 = \mathbf{v}^{10} + c^{11}\mathbf{V}^1 \tag{2.6}$$

Here v^{10} is a solution of problem (2.2) with the forces $\mathbf{T}^+ = \mathbf{T}^- = {}^1/_2\mathbf{T}^1$ concentrated at the points $\mathbf{P}^{\pm}$ and c^{11} is a constant to be determined. (In (2.6) and in what follows, the rigid displacements Ω_0 and Ω_ε are not considered).

Following /6/ we find that the vector function (2.6) can be represented as follows near the point $(b_+, +0)$:

$$\mathbf{v}^1(\mathbf{x}) = \sum_{j=1}^{4} l_{1j}^+ \mathbf{U}^j(\mathbf{y}) + l_{15}^+ \mathbf{S}^{(1)}(\mathbf{y}) + l_{06}^+ \boldsymbol{\Upsilon}^1(\mathbf{y}) + O(|\mathbf{y}|^2 \ln |\mathbf{y}|) \tag{2.7}$$

$$\Upsilon_r^1(r, \theta) = -(2\pi\mu)^{-1} r \{\ln r (\cos 2\theta + {}^1/_2(\varkappa - 1)) - \theta \sin 2\theta - {}^1/_4(\varkappa + 1)\}$$
$$\Upsilon_\theta^1(r, \theta) = (2\pi\mu)^{-1} r \{\ln r \sin 2\theta + \theta \cos 2\theta - {}^1/_2\theta(\varkappa + 1)\} \tag{2.8}$$

Here $S^{(1)}$ is the first column of the Somigliana tensor, (r, θ) are polar coordinates corresponding to $\mathbf{y}$; l_{1j}^+ are certain constants calculated from formulas analogous to (1.9), the coefficient l_{15}^+ and the similar coefficient l_{15}^- have the following form when expanded near $(-b_-, 0)$:

$$l_{15}^\pm = {}^1/_2 T_1^1 \pm c^{11} \tag{2.9}$$

Let us now construct the next term $\varepsilon^2 \mathbf{v}^2$ of the asymptotic form of the vector $\mathbf{u}^\varepsilon$. By virtue of (2.1) and of the similar formula for $\mathbf{v}^1$ we find that $\mathbf{v}^2$ satisfies (2.2) in which the load $\mathbf{q}^\pm$ is determined by (2.3), and

$$q_j^+(x_1) = -\sigma_{j2}(\mathbf{v}^1_{,2}; x_1, +0) - {}^1/_2 \sigma_{j2}(\mathring{\mathbf{v}}_{,22}; x_1, +0), \; x_1 \in [-b_-, b_+] \tag{2.10}$$

According to (2.7) the quantity (2.10) has singularities of the order of $|x_1 \mp b_\pm|^{-1}$. This implies that problem (2.2) for $\mathbf{v}^2$ should also be solved in the class of vector functions with singularities at the points $\mathbf{P}^\pm$. It will be shown below that it is necessary to allow the singularity $O(|\mathbf{x} - \mathbf{P}^\pm| \times |\ln|\mathbf{x} - \mathbf{P}^\pm||)$. The solutions belonging to this class are determined with an accuracy up to a linear combination of three vectors. These are, firstly, the solution $\mathbf{V}^1$ introduced above, and secondly the solutions $\mathbf{V}^2$ and $\mathbf{V}^3$ of the problem with concentrated forces $\mathbf{T}^\pm = (0, \pm 1)$ and moments $R^+ = -(b_+ + b_-)$, $R^- = 0$ or $R^+ = 0$, $R^- = -(b_+ + b_-)$, respectively. Thus

$$\mathbf{v}^2 = \mathbf{v}^{20} + c^{21}\mathbf{V}^1 + c^{22}\mathbf{V}^2 + c^{23}\mathbf{V}^3 \tag{2.11}$$

Here c^{2j} are constants to be determined. Using the results of /6/, we choose the solution $\mathbf{v}^{20}$ so that the asymptotic form of the sum (2.11) has the following form:

$$\mathbf{v}^2(\mathbf{y}) = l_{20}^+ \mathbf{S}^{(0)}(\mathbf{y}) + l_{21}^+ \mathbf{U}^1(\mathbf{y}) + l_{22}^+ \mathbf{U}^2(\mathbf{y}) + l_{23}^+ \mathbf{S}^{(1)}(\mathbf{y}) + l_{24}^+ \mathbf{S}^{(2)}(\mathbf{y}) + l_{06}^+ \boldsymbol{\Upsilon}^2(\mathbf{y}) + l_{15}^+ \boldsymbol{\Upsilon}^\circ(\mathbf{y}) + O(|\mathbf{y}|(\ln|\mathbf{y}|)^2) \tag{2.12}$$

$$S^{(1)} = [-4\mu\pi]^{-1} \{\ln r [1 + \varkappa](\cos\theta, -\sin\theta) + [\varkappa - 1]\theta(\sin\theta, \cos\theta) - 2(0, \sin\theta)\}$$
$$S^{(2)} = [-4\mu\pi]^{-1} \{\ln r [1 + \varkappa](\sin\theta, \cos\theta) + [1 - \varkappa]\theta(\cos\theta, -\sin\theta) + 2(0, \cos\theta)\}$$
$$S^{(0)} = [4\mu\pi r]^{-1}([1 + \varkappa]\sin 2\theta, [\varkappa - 1]\cos 2\theta + 2) \tag{2.13}$$
$$(\Upsilon_r^2(r, \theta), \Upsilon_\theta^2(r, \theta)) = [4\mu\pi^2]^{-1}\{[1 + \varkappa][\ln^2 r + 2\varkappa^{-1}\ln r - \theta^2] \cdot (\cos\theta, -\sin\theta) + 2\theta[\ln r + \varkappa^{-1} + 1](\sin\theta, \cos\theta) + (0, 4\theta\cos\theta - 4[\ln r + \varkappa^{-1}]\sin\theta)\}$$

$$(\Upsilon_r^\circ(r, \theta), \Upsilon_\theta^\circ(r, \theta)) = [2\mu\pi^2 r]^{-1}\{\ln r([1 + \varkappa]\cos 2\theta, [1 - \varkappa]\sin 2\theta) + \theta([1 + \varkappa]\sin 2\theta \; [\varkappa - 1]\cos 2\theta) + (1, -2\varkappa[1 + \varkappa]^{-1}\sin 2\theta)\}$$

Below we shall find it useful to use the following formulas for the coefficients $l_{20}^+, l_{23}^+, l_{24}^+$ and for the analogous coefficients in the expansion of $\mathbf{v}^2$ near $(-b_-, 0)$:

$$l_{23}^\pm = {}^1/_2 T_1^2 \pm c^{21}, \quad l_{24}^\pm = {}^1/_2 T_2^2 \pm c^{22} \pm c^{23} \tag{2.14}$$
$$l_{20}^+ = R^2 - {}^1/_2 T_2^2 (b_+ + b_-) - c^{23}(b_+ + b_-), \quad l_{20}^- = {}^1/_2 T_2^2 (b_+ + b_-) - c^{22}(b_+ + b_-)$$

Let us determine the unknown T_1^2, T_2^2, R^2 in (2.14) using the method given in /7/. We denote by $\Omega(\delta)$ the region Ω_0 from which the sets $\{\mathbf{x} : |\mathbf{x} - \mathbf{P}^\pm| < \delta, \; x_2 > 0\}$ have been removed, and substitute into the Betti formula for the region $\Omega(\delta)$ the vectors $\mathbf{v}^2$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $(-x_2, x_1)$. Taking (2.10), (2.12) and (2.4) into account, applying a transformation similar to (2.5) and passing to the limit as $\delta \to 0$, we obtain the following relations:

$$0 = \pi^{-1}(2\varkappa^{-1} - 1)(l_{06}^+ - l_{06}^-) + {}^1/_2(l_{05}^+ - l_{05}^-) - (l_{23}^+ + l_{23}^-) \tag{2.15}$$
$$0 = {}^1/_2(l_{06}^+ - l_{06}^-) - (l_{24}^+ + l_{24}^-)$$
$$0 = {}^1/_2(b_+ + b_-) l_{06}^+ - {}^1/_2(l_{04}^+ - l_{04}^-) - l_{24}^+(b_+ + b_-) - (l_{20}^+ + l_{20}^-)$$

From (2.5) and (2.14) we obtain

$$T_1^2 = -\pi^{-1}(1-2\varkappa^{-1})(l_{06}^+ - l_{06}^-) + {}^1/_2(l_{05}^+ - l_{05}^-),\ T_2^2 = {}^1/_2(l_{06}^+ - l_{06}^-)$$
$$R^2 = {}^1/_4(l_{06}^+ + l_{06}^-)(b_+ + b_-) - {}^1/_2 T_1^1$$

3. ***The boundary layer near the points*** $\mathbf{P}\pm$. We have obtained in Sect.2 the first terms of the asymptotic solution of problem (1.1)-(1.3) away from the crack N_ε. A boundary layer appears near the tips of the crack N_ε. Since the problem is symmetrical, it is sufficient to consider only the right-hand tip. Let us write (1.1) and (1.3) in "rapid" variables $\xi_1 = \varepsilon^{-1}(x_1 - b_+)$, $\xi_2 = \varepsilon^{-1}x_2$ and put $\varepsilon = 0$. The region $\Omega_\varepsilon^+ = \{x \in \Omega_\varepsilon : x_2 > 0\}$ will be transformed into a half-plane $\mathbf{R}_+^2$ with a ray $\Xi = \{\xi : \xi_2 = 1,\ \xi_1 \leqslant 0\}$, cut away, and problem (1.1), (1.3) will become

$$L(\partial/\partial\xi)\mathbf{Z}(\xi) = 0,\quad \xi \in \mathbf{R}_+^2 \setminus \Xi;\quad \sigma_{j2}(\mathbf{Z};\xi) = 0,\quad \xi \in \partial\mathbf{R}_+^2 \cup \Xi,\quad j = 1, 2 \tag{3.1}$$

The set $\mathbf{R}_+^2 \setminus \Xi$ has two "exits" to infinity, one in the form of an angle, and the other in the form of a half-strip. We shall indicate the solutions of problem (3.1) which show, at infinity, not more than a logarithmic growth in the angle, and not more than a power growth in the half-strip. According to the general results /6-8/ there are precisely five linearly independent solutions of this type. Two of them are trivial: $\mathbf{Z}^j(\xi) = \mathbf{e}^j$, $j = 1, 2$ are the rigid translational displacements with unit vectors in $\mathbf{R}^2$. The three remaining ones correspond to transverse and longitudinal forces and a moment, applied at infinity in the half-strip, and they have the following asymptotic form:

$$\mathbf{Z}^3(\xi) = -{}^1/_{12}D^{-1}\mathbf{Y}^{(1,1)}(\xi) - {}^1/_2 D^{-1}\mathbf{Y}^{(2,2)}(\xi) + m_1(-\xi_2, \xi_1) + O(1) \tag{3.2}$$
$$\mathbf{Z}^4(\xi) = D^{-1}\mathbf{Y}^{(2,3)}(\xi) + m_2(-\xi_2, \xi_1) + O(1)$$
$$\mathbf{Z}^5(\xi) = -D^{-1}\mathbf{Y}^{(2,2)}(\xi) + m_3(-\xi_2, \xi_1) + O(1)$$

$$\mathbf{Z}^3(\xi) = S^{(1)}(\xi) + \Upsilon^\circ(\xi) + O(|\xi|^{-2});\quad \mathbf{Z}^4(\xi) = S^{(2)}(\xi) + O(|\xi|^{-2})$$
$$\mathbf{Z}^5(\xi) = S^{(0)}(\xi) + O(|\xi|^{-2});\quad D = [3(\lambda + 2\mu)]^{-1}(\lambda + \mu)\mu \tag{3.3}$$
$$\mathbf{Y}^{(i,k)}(\xi) = \sum_{j=0}^{k}(j!)^{-1}\xi_1^{\,j}\mathbf{X}^{(i,k-j)}(\xi_2);\quad i = 1, 2;\quad k = 0, 1, \ldots, 2i - 1$$

$$\mathbf{X}^{(1,0)}(\xi_2) = \mathbf{e}^1,\ \mathbf{X}^{(2,0)}(\xi_2) = \mathbf{e}^2,\ \mathbf{X}^{(1,1)}(\xi_2) = -(\lambda + 2\mu)^{-1}\lambda(\xi_2 - {}^1/_2)\mathbf{e}^2 \tag{3.4}$$
$$\mathbf{X}^{(2,1)}(\xi_2) = -(\xi_2 - {}^1/_2)\mathbf{e}^1,\ \mathbf{X}^{(2,2)}(\xi_2) = [2(\lambda + 2\mu)]^{-1}\lambda((\xi_2 - {}^1/_2)^2 - {}^1/_{12})\mathbf{e}^2$$
$$\mathbf{X}^{(2,3)}(\xi_2) = [6(\lambda + 2\mu)]^{-1}\{(3\lambda + 4\mu)(\xi_2 - {}^1/_2)^3 - {}^1/_4(11\lambda + 12\mu)(\xi_2 - {}^1/_2)\}\mathbf{e}^1$$

In (3.2) $\xi_1 \to -\infty, \xi_2 < 1$, while in (3.3) $|\xi| \to \infty, \xi_2 > 1$; m_i are constants and $S^{(j)}$ are columns in (2.13).

If we allow the linear growth of solutions in the angle, then another two vectors $\mathbf{U}^3$ and $\mathbf{U}^4$ will appear, satisfying the homogeneous problem (3.1). Passing to the quadratic growth produces two new solutions. One of these solutions will be the vector $\mathbf{U}^5$, and the other solution has the following asymptotic form:

$$\mathbf{Z}^6(\xi) = \mathbf{U}^6(\xi) + \Upsilon^1(\xi) + \Upsilon^2(\xi) + O(|\xi|^{-1}(\ln|\xi|)^2)(\xi_2 > 1, |\xi| \to \infty) \tag{3.5}$$

The expansion of $\mathbf{Z}^6$ at infinity in the half-strip contains a linear combination of vectors (3.4). In what follows we shall only use the coefficient accompanying the term $\mathbf{Y}^{(2,3)}$. The coefficient is found by applying the Betti formula in the region

$$\{\xi \in \mathbf{R}_+^2 \setminus \Xi : |\xi| < R\} \cup \{\xi : 0 < \xi_2 < 1, 0 > \xi_1 > -R\}$$

for the vectors $\mathbf{Z}^6$ and $\mathbf{e}^2$ (see /7/ and analogous arguments in /5/). Calculating the contour integrals with help of the asymptotic representation of $\mathbf{Z}^6$ and passing to the limit as $R \to \infty$, we obtain the asymptotic form

$$\mathbf{Z}^6(\xi) = -{}^1/_2 D^{-1}\mathbf{Y}^{(2,3)}(\xi) + O(|\xi_1|^2)\quad (\xi_1 \to -\infty, 0 < \xi_2 < 1) \tag{3.6}$$

Let us obtain the first three terms of the expansion in the boundary layer

$$\mathbf{u}^\varepsilon(\mathbf{x}) \sim \mathbf{Z}^{0+}(\xi) + \varepsilon\mathbf{Z}^{1+}(\xi) + \varepsilon^2\mathbf{Z}^{2+}(\xi),\quad |\mathbf{y}| < 2\varepsilon^{1/2} \tag{3.7}$$

Taking into account the conditions of matching this representation with the expansion $\mathbf{u}^\varepsilon(\mathbf{x}) \sim \mathbf{v}^\circ(\mathbf{x}) + \varepsilon\mathbf{v}^1(\mathbf{x}) + \varepsilon^2\mathbf{v}^2(\mathbf{x})$ at $|\mathbf{y}| > {}^1/_2\varepsilon^{1/2}$, we obtain, from (3.3), (3.5) and (1.8), (2.7) and (2.12), the relations

$$\mathbf{Z}^{\circ+}(\xi) = l_{01}^{+}\mathbf{e}^1 + l_{02}^{+}\mathbf{e}^2, \quad \mathbf{Z}^{1+}(\xi) = l_{03}^{+}\mathbf{U}^3(\xi) + l_{04}^{+}\mathbf{U}^4(\xi) + l_{15}^{+}\mathbf{Z}^3(\xi) + (l_{11}^{+} + l_{15}^{+}(4\mu\pi)^{-1}(1+\varkappa)\ln\varepsilon)\,\mathbf{e}^1 + l_{12}^{+}\mathbf{e}^2 + l_{20}^{+}\mathbf{Z}^5(\xi)$$
$$\mathbf{Z}^{2+}(\xi) = l_{05}^{+}\mathbf{U}^5(\xi) + l_{06}^{+}\mathbf{Z}^6(\xi) + l_{13}^{+}\mathbf{U}^3(\xi) + (l_{14}^{+} - l_{06}^{+}2\pi^{-1}\ln\varepsilon)\,\mathbf{U}^4(\xi) + (l_{23}^{+} + l_{06}^{+}2\pi^{-1}\ln\varepsilon)\,\mathbf{Z}^3(\xi) + l_{24}^{+}\mathbf{Z}^4(\xi) + ((l_{23}^{+} + 2(\pi\varkappa)^{-1}l_{06}^{+})(4\mu\pi)^{-1}(1+\varkappa)\times \ln\varepsilon + l_{21}^{+} + l_{06}^{+}(4\mu\pi^2)^{-1}(1+\varkappa)\ln^2\varepsilon)\,\mathbf{e}^1 + (l_{22}^{+} + l_{24}^{+}(4\mu\pi)^{-1}(1+\varkappa))\,\mathbf{e}^2 + \ldots \tag{3.8}$$

Here the repeated dots denote the term const $\mathbf{Z}^5$, and the constant can be found only after obtaining the solution $\mathbf{v}^{3\circ}$ (in exactly the same manner the constant l_{20}^{+}, appearing in the expansion of $\mathbf{v}^{2\circ}$, occurs in the boundary layer $\mathbf{Z}^{1+}$, see also Sect.7 below). It will be shown that this constant has no significant effect on the SIF, and the above statement applies with equal force to many other terms from (3.8), and in particular to terms which contain $\ln\varepsilon$. Formulas analogous to (3.8) hold for the boundary layer $\mathbf{Z}^{\circ-} + \varepsilon\mathbf{Z}^{1-} + \varepsilon^2\mathbf{Z}^{2-} + \ldots$, corresponding to the point $(-b_-, 0)$.

4. The asymptotic form of the solution in a thin strip. Using the well-known algorithm for constructing the asymptotic solutions of elliptical problems in thin regions (/4, 10, 9/ et al), we shall write the solution $\mathbf{u}^\varepsilon(\mathbf{x})$ on the set $(-b_-, b_+) \times (0, \varepsilon)$ in the form

$$\mathbf{u}^\varepsilon(\mathbf{x}) \sim \sum_{j=-1}^{\infty} \varepsilon^j (\mathbf{w}^j(x_1) + \varepsilon\mathbf{W}^j(x, \zeta)), \quad \zeta = \varepsilon^{-1}x_2 \tag{4.1}$$

In what follows, we shall make use of the first few terms of the series, namely of

$$w_1^{-1}(x_1) = 0, \quad w_2^{-1}(x_1) = w_2(x_1), \quad w_1^{\circ}(x_1) = w_1(x_1)$$
$$\mathbf{W}^{-1}(x_1, \zeta) = (\partial w_2/\partial x_1)\,\mathbf{X}^{(2,1)}(\zeta), \quad \mathbf{W}^{\circ}(x_1, \zeta) = (\partial w_1/\partial x_1)\,\mathbf{X}^{(1,1)}(\zeta) + (\partial^2 w_2/\partial x_1^2)\,\mathbf{X}^{(2,2)}(\zeta), \quad \mathbf{W}^1(x_1, \zeta) = (\partial^3 w_2/\partial x_1^3)\,\mathbf{X}^{(2,3)}(\zeta) \tag{4.2}$$

We have omitted from (4.2) the terms which depend on the components of the functions w_2°, w_j^1, w_j^2 (they can also be expressed in terms of the elements of the Jordan chains (3.4)) and the function w_j satisfies the relations

$$(\partial^2 w_1/\partial x_1^2)(x_1) = 0, \quad (\partial^4 w_1/\partial x_1^4)(x_1) = 0, \quad x_1 \in (-b_-, b_+) \tag{4.3}$$

The expansion (4.1) must be combined with the solutions of the boundary layer-type constructed in Sect.3. According to (3.2), (3.6)-(3.8) the following formulas hold for $\xi_1 < 2\varepsilon^{-1/2}$, $0 < \xi_2 < \varepsilon$:

$$\mathbf{Z}^{\circ+}(\xi) + \varepsilon\mathbf{Z}^{1+}(\xi) + \varepsilon^2\mathbf{Z}^{2+}(\xi) = l_{01}^{+}\mathbf{e}^1 + \varepsilon l_{04}^{+}U^4(\xi) + \varepsilon l_{15}^{+}(-(12D)^{-1}\mathbf{Y}^{(1,1)}(\xi) - (2D)^{-1}\mathbf{Y}^{(2,2)}(\xi)) + \varepsilon l_{20}^{+}(-D)^{-1}Y^{(2,2)}(\xi) + \varepsilon^2\mathbf{Y}^{(2,3)}(\xi)(-(2D)^{-1}l_{06}^{+} + D^{-1}l_{24}^{+}) + \ldots$$

The repeated dots denote the terms in the asymptotic form which are disregarded when considering the terms w_1 and w_2 in (4.1).

Using the formula $\mathbf{U}^4 = (12D)^{-1}\mathbf{Y}^{(1,1)}$ in relation (4.2) we conclude that for the asymptotic representations (4.1) and (3.7) to be identical in the intermediate zone $\xi_1 = O(\varepsilon^{-1/2})$ (or $x_1 = b_+ + O(\varepsilon^{1/2})$), the following relations must hold:

$$w_2(b_+) = 0, \quad (\partial w_2/\partial x_1)(b_+) = 0 \quad (\partial^2 w_2/\partial x_1^2)(b_+) = -D^{-1}(1/2\,l_{15}^{+} + l_{20}^{+})$$
$$(\partial^3 w_2/\partial x_1^3)(b_+) = D^{-1}(l_{24}^{+} - 1/2\,l_{06}^{+}); \quad w_1(b_+) = l_{01}^{+} \quad (\partial w_1/\partial x_1)(b_+) = (12D)^{-1}(l_{04}^{+} - l_{15}^{+}) \tag{4.4}$$

Considering now the boundary layer corresponding to the point $(-b_-, 0)$, we obtain

$$w_2(-b_-) = 0 \quad (\partial w_2/\partial x_1)(-b_-) = 0 \quad (\partial^3 w_2/\partial x_1^2)(-b_-) = D^{-1}(1/2\,l_{15}^{-} + l_{20}^{-})$$
$$(\partial^3 w_2/\partial x_1^3)(-b_-) = -D^{-1}(l_{24}^{-} + 1/2\,l_{06}^{-}); \quad w_1(-b_-) = l_{01}^{-}, \quad (\partial w_1/\partial x_1)(-b_-) = (12D)^{-1}(l_{04}^{-} + l_{15}^{-}) \tag{4.5}$$

We have, in formulas (4.4) and (4.5), the Dirichlet data at the point $x_1 = \pm b_{\pm}$ for Eq.(4.3). Solving the boundary value problems we find, that

$$w_1(x) = (b_+ + b_-)^{-1}(v_1^{\circ}(b_+, +0) - v_1^{\circ}(-b_-, +0))(x_1 - b_+) + v_1^{\circ}(b_+, 0), \quad w_2 = 0 \tag{4.6}$$

The remaining six equations in (4.4) and (4.5) comprise a system of linear algebraic equations for determining the unknown constants c^{11}, c^{22}, c^{23} from the representations (2.9) and (2.14). We find that by virtue of (2.15) this overdefined system has the solution

$$\begin{gathered} c^{11}={}^1/_2(\sigma_{11}(\mathbf{v}^\circ; b_+, +0)+\sigma_{11}(\mathbf{v}^\circ; -b_-, +0))- \\ 12D(b_++b_-)^{-1}(v_1{}^\circ(b_+, +0)-v_1{}^\circ(-b_-, +0)) \\ c^{22}={}^1/_4(\sigma_{11,1}(\mathbf{v}^\circ; b_+, +0)-\sigma_{11,1}(\mathbf{v}^\circ; -b_-, +0))-{}^1/_2(b_++b_-)^{-1}\times \\ [12D(b_++b_-)^{-1}(v_1{}^\circ(b_+, +0)-v_1{}^\circ(-b_-, +0))-\sigma_{11}(\mathbf{v}^\circ; -b_-, +0)], \\ c^{23}={}^1/_2\sigma_{11,1}(\mathbf{v}^\circ; -b_-, +0)-{}^1/_2(b_++b_-)^{-1}[12D(b_++b_-)^{-1}(v_1{}^\circ(b_+, +0)- \\ v_1{}^\circ(-b_-, +0))-\sigma_{11}(\mathbf{v}^\circ, -b_-, +0)] \end{gathered} \tag{4.7}$$

Thus the vectors (2.6), (2.11) and (3.7) as well as the principal terms of the series (4.1) are all completely determined.

5. *The asymptotic form of the SIF.* The asymptotic form of the solution of problem (1.1)-(1.3) obtained in the previous section enables us, as in /4, 5, 11, 12/, to find the SIF of $K_j^\pm(\varepsilon)$ and $k_j^\pm(\varepsilon)$ at the tips of the cracks M and N_ε. Since outside the neighbourhood of the crack N_ε the solution of the initial problem can be written in the form $\mathbf{v}^\circ(\mathbf{x})+\varepsilon\mathbf{v}^1(\mathbf{x})+\ldots$ (see Sect.2), it follows that

$$K_j^\pm(\varepsilon)=K_j^\pm+\varepsilon(K_{j,1}^\pm+c^{11}F_j^\pm)+O(\varepsilon^2),\quad j=1,2 \tag{5.1}$$

Here $K_j^\pm(\varepsilon)$, $K_j^\pm$, $K_{j,1}^\pm$ and $F_j^\pm$ are the coefficients of $r_\pm^{1/2}$ in expansions of the fields $\mathbf{u}^\varepsilon$, $\mathbf{v}^\circ$, $\mathbf{v}^{\circ 1}$ and $\mathbf{V}^1$, of the form (1.6), and the constant c^{11} is given by (4.7).

According to Sect.3 the field $\mathbf{u}^\varepsilon(\mathbf{x})$ is asymptotically equal, in a small neighbourhood of the point $\mathbf{P}\pm$, to the sum $\mathbf{Z}^{0+}(\xi)+\varepsilon\mathbf{Z}^{1+}(\xi,\ln\varepsilon)+\ldots$. Analysing formulas (3.8) for the vector functions $\mathbf{Z}^{0+}$ and $\mathbf{Z}^{1+}$ we conclude that $l_{15}{}^+\mathbf{Z}^3(\xi)$ and $l_{20}{}^+\mathbf{Z}^5(\xi)$ are the only terms which are not smooth.

The special solutions $\mathbf{Z}^k$ $(k=3,4,5)$ near the tips of the cut Ξ can be represented in the form

$$\mathbf{Z}^k(\xi)=\mathbf{c}^k+\rho^{1/2}(F_{k,1}\boldsymbol{\Phi}^1(\varphi)+F_{k,2}\boldsymbol{\Phi}^2(\varphi))+O(\rho),\quad \rho\to 0 \tag{5.2}$$

Here $\mathbf{c}^k$ is a constant vector, (ρ,φ) are polar coordinates with centre at the tip of the ray Ξ, and the edges of Ξ are given by the equations $\varphi=\pm\pi$. Explicit expressions for the SIF were obtained in /13/ in expansions (5.2) of the special solutions $\mathbf{Z}^3-\mathbf{Z}^5$, $\mathbf{Z}^4$, $\mathbf{Z}^5$ of the homogeneous problem (3.1). Using the notation adopted in /13/, we can write the vectors $\mathbf{F}_k=(F_{k1},F_{k2})$ in the form

$$\mathbf{F}_3=\mathbf{K}_T+{}^1/_2\mathbf{K}_M,\quad \mathbf{F}_4=\mathbf{K}_N,\quad \mathbf{F}_5=\mathbf{K}_M \tag{5.3}$$

$$\mathbf{K}_M\approx(1.932,\ -1.506),\quad \mathbf{K}_T\approx(0.4346,\ 0.05578),\quad \mathbf{K}_N\approx(1.951,\ 0.032)$$

Thus, according to (5.2), (5.3) and (2.9), (2.14) and (4.7) we have

$$\begin{gathered} k_j^\pm(\varepsilon)=\mp\varepsilon^{1/2}(12D(v_1{}^\circ(b_+, +0)-v_1{}^\circ(-b_-, +0))(b_++b_-)^{-1}- \\ \sigma_{11}(\mathbf{v}^\circ;\pm b_\pm, +0)(F_{3j}^\pm-{}^1/_2F_{5j}^\pm)+O(\varepsilon^{3/2}|\ln\varepsilon|),\quad \mathbf{F}_k^\pm=(\pm F_{k1}, F_{k2}) \\ (k=3.5) \end{gathered} \tag{5.4}$$

The principal terms of the asymptotic form of $k_j^\pm(\varepsilon)$ in (5.4) is expressed by the solution of the problem (1.4), (1.5) in the region with a single crack. This makes it possible to determine, in many problems, the SIF at the tips of the smaller crack N_ε without additional computations. We stress that the beam approximation in the theory of cracks (/14, 15/ et al.) or the Cherepanov-Rice integral (/14, 17, 16/) et al.) make it possible for the problems in canonical domains to find the sum $I^\pm(\varepsilon)=k_1^\pm(\varepsilon)^2+k_2^\pm(\varepsilon)^2$, while $k_1^\pm(\varepsilon)$ and $k_2^\pm(\varepsilon)$ cannot be determined separately in this manner. According to (5.3) and (5.4), $k_1^\pm(\varepsilon)k_2^\pm(\varepsilon)^{-1}\approx 0.779$, therefore the values of $k_j^\pm(\varepsilon)$ are obtained from $I^\pm(\varepsilon)$ (apart from the sign).

6. *Examples.* 1°. Let us consider the plane $\Omega=\mathbf{R}^2$ with cracks M and N_ε, under a uniaxial tension of intensity p°, applied at an angle α to the x_1 axis. Using the explicit solution of the problem of a plane with a single crack /18/, we can write formulas (5.4) for the SIF at the tips of the cracks N_ε in the following specific form:

$$\begin{gathered} k_j^\pm(\varepsilon)\sim\varepsilon^{1/2}\cdot 2p_{sc}(F_{3j}^\pm-{}^1/_2F_{5j}^\pm)(\mp B^{-1}(A_+R_+-A_-R_-)-b_\pm A_\pm R_\pm) \\ B=b_++b_-,\ R_\pm=(a\mp b_\pm)^{1/2},\ A_\pm=(a\pm b_\pm)^{1/2},\ p_{sc}=p^\circ\sin\alpha\cos\alpha \end{gathered} \tag{6.1}$$

Moreover, in (2.4) $q_1^+(x_1) = -2p_{sc}a^2(a^2 - x_1^2)^{-3/2}$, $q_2^+(x_1) = 0$, and this means that according to the integral representation of the SIF /18/ formula (5.1) will take the form

$$K_1^{\pm}(\varepsilon) \sim p^\circ \sin^2\alpha\,(\pi a)^{1/2},\quad K_2^{\pm}(\varepsilon) \sim p_{sc}\left((\pi a)^{1/2} - {}^1/_2\varepsilon\,(\pi a)^{-1/2}G^{\pm}\right) \tag{6.2}$$
$$G^+ = \ln(A_+R_-A_-^{-1}R_+^{-1} + aB\,(2\,(R_+R_-)^2) - (A_+A_-R_+R_-)^{-1}) + 2B^{-1}(R_+R_-)^{-1}H)$$
$$G^- = \ln(A_+R_+A_-^{-1}R_-^{-1}) + aB\,(2\,(A_+A_-)^{-1} - (A_+A_-R_+R_-)^{-1}) - 2B^{-1}(A_+A_-)^{-1}H)$$
$$H = (A_+^2 + A_-^2)\,R_+R_- - (R_+^2 + R_-^2)\,A_+A_-$$

2°. The asymptotic analysis carried out in the previous sections can also be applied to the problem of a plane with a semi-infinite cut $M = \{x \in \mathbf{R}^2 : x_2 = 0, x_1 \leqslant a\}$ and a finite crack N_ε. Let the loading be applied in such a manner that the solution $\mathbf{u}^\varepsilon$ will have the following asymptotic form at infinity:

$$\mathbf{u}^\varepsilon(\mathbf{x}) = r^{1/2}(C_1\Phi^1(\theta) + C_2\Phi^2(\theta)) + O(r^{-1/2})\quad (r\to\infty) \tag{6.3}$$

Here (r, θ) are polar coordinates with centre $(a, 0)$, Φ^j are the angular parts of (1.7), and C_1 and C_2 are load parameters. In other words, we shall consider a large crack with a small crack N_ε situated near its tip. In this case $\mathbf{v}^\circ$ will be indentical with the expression singled out in (6.3), and according to (5.4) and (5.1) we shall have

$$k_j^{\pm}(\varepsilon) \sim \mp\varepsilon^{1/2}2C_2(2\pi)^{-1/2}(F_{3j}^{\pm} - {}^1/_2F_{5j}^{\pm})(R_{\pm}^{-1} + 2B^{-1}(R_+ - R_-))$$
$$K_1^+(\varepsilon) \sim C_1,\ K_2^+(\varepsilon) \sim C_2 + 2\varepsilon C_2\pi^{-1}B^{-1}R_+^{-1}R_-^{-1}(R_+ + R_-)^2$$

3°. The method of computing SIF can also be used in the case of a crack situated at a short distance from the boundary. We shall consider, as an example, the problem of the action of concentrated normal force Q on the half-plane $\Omega = \{\mathbf{x}: x_2 > 0\}$ (we assume that $b_{\pm} > 0$ and that the force is applied at the point $\mathbf{x} = 0$). Since we assumed earlier that the crack edges are stress-free, it follows that the algorithm given in Sect.2 and 3 needs some modification. We will seek the asymptotic form of the solution in the thin strip $(-b_-, b_+) \times (0, \varepsilon)$ in the form

$$\mathbf{u}^\varepsilon(\mathbf{x}) \sim \varepsilon^{-3}w_2(x_1)\,\mathbf{X}^{(2,0)} + \varepsilon^{-2}(\partial w_2/\partial x_1)(x_1)\,\mathbf{X}^{(2,1)} + \varepsilon^{-1}(\partial^2 w_2/\partial x_1^2)(x_1)\,\mathbf{X}^{(2,2)} + \tag{6.4}$$
$$(\partial^3 w_2/\partial x_1^3)(x_1)\,\mathbf{X}^{(2,3)} + \ldots,\ w_2(x_1) = \gamma_0 + \gamma_1x_1 + \gamma_2x_1^2 + \gamma_3x_1^3 + {}^1/_6\,QD^{-1}x_1^3\Theta(x_1)$$

Here Θ is the Heaviside function and repeated dots denote unimportant terms. The boundary layers $\mathbf{Z}^{\pm}$ near $\mathbf{P}^{\pm}$ are given by the formulas

$$\mathbf{Z}^{\pm}(\xi) = \varepsilon^{-3}w_2(\pm b_{\pm})\mathbf{U}^2(\xi) + \varepsilon^{-2}(\partial w_2/\partial x_1)(\pm b_{\pm})\,\mathbf{U}^3(\xi) + \varepsilon^{-1}(\partial^2 w_2/\partial x_1^2)(\pm b_{\pm})\,\mathbf{Z}^3(\xi) + \tag{6.5}$$
$$(\partial^3 w_2/\partial x_1^3)(\pm b_{\pm})\,\mathbf{Z}^4(\xi)$$

From the matching condition it follows that the first coefficient in the representation $\mathbf{u}^\varepsilon(\mathbf{x}) \sim \varepsilon^{-3}\mathbf{v}_0(\mathbf{x}) + \varepsilon^{-2}\mathbf{v}^1(\mathbf{x}) + \varepsilon^{-1}\mathbf{v}^2(\mathbf{x}) + \mathbf{v}^3(\mathbf{x})$ of the solution away from the crack satisfies the relation $\mathbf{v}^\circ(\pm b_{\pm}) = w_2(\pm b_{\pm})$, i.e. $\mathbf{v}^\circ$ is a rigid displacement which can be made equal to zero. Similarly, $\mathbf{v}^1(\pm b_{\pm}) = (\partial w_2/\partial x_1)(\pm b_{\pm}) = 0$. The constants γ_j in (6.4) are found from the conditions indicated, and have the form

$$\gamma_0 = {}^1/_3\gamma b_+^3b_-^3,\ \gamma_1 = {}^1/_2\gamma b_+^2b_-^2(b_+ - b_-),\ \gamma_2 = -\gamma b_+^2b_-^2 \tag{6.6}$$
$$\gamma_3 = -{}^1/_6\gamma b_+^2(b_+ + 3b_-),\ \gamma = QD^{-1}(b_+ + b_-)^{-3}$$

Thus, in accordinace with (6.4), (6.5) and (5.2), the SIF at the tips of the crack N_ε are given by

$$k_j^{\pm}(\varepsilon) = \varepsilon^{-3/2}\gamma\,(b_+ + b_-)\,b_{\pm}b_{\mp}^2F_{5j} + O(\varepsilon^{-1/2}) \tag{6.7}$$

4°. In the cases when $\alpha = {}^1/_2\pi$ or $C_2 = 0$, the asymptotic formulas (6.1) or (6.4) take the form $k_j^{\pm}(\varepsilon) = O(\varepsilon^{3/2})$ and become lacking in content (when $\alpha = 0$, the terms of the asymptotic form separated in (6.1) and (6.2) also vanish as the loading takes place along the cracks and all SIF are equal to zero). Both situations are characterized by the fact that $\sigma_{11}(\mathbf{v}^\circ) = 0$ at the edges of the crack M. If we construct the next term of the asymptotic form of the solution $\mathbf{u}^\varepsilon(\mathbf{x})$ (see Sect.7), the asymptotic form of the SIF at the tips of the crack N_ε will take the following form for the problems discussed in 1° and 2° respectively:

$$k_j^{\pm}(\varepsilon) \sim \mp\varepsilon^{3/2}(a^2(A_{\pm}R_{\pm})^{-3}F_{3j}^{\pm} - \{{}^7/_8a^2(A_{\pm}R_{\pm})^{-3} + B^{-2}(A_+R_+ - A_-R_-) + \tag{6.8}$$
$${}^1/_3B^{-1}(\pm b_{\pm}(A_{\pm}R_{\pm})^{-1} + b_+(A_+R_+)^{-1} - b_-(A_-R_-)^{-1}\}F_{5j}^{\pm})$$
$$k_j^{\pm}(\varepsilon) \sim \pm\varepsilon^{3/2}\,{}^1/_4C_1(2\pi)^{-1/2}\{R_{\pm}^{-3}F_{3j}^{\pm} - ({}^7/_6R_{\pm}^{-3} \mp 8B^{-2}(R_+ - R_-))F_{5j}^{\pm}\}$$

7. Asymptotic form of the solution in a special case. We shall assume that $\sigma_{11}(\mathbf{v}^\circ) = 0$ on M_+. Then, in accordance with (2.2), $q_j^+ = -\sigma_{j2,2}(\mathbf{v}^\circ) = \sigma_{j1,1}(\mathbf{v}^\circ) = 0$, i.e. the solution $\mathbf{v}^1$ of problem (2.2) will represent a rigid displacement which can be assumed equal to zero. Let us change the upper limit of summation in (1.8) to 8, and

$$\begin{gathered} l_{04}^+ = l_{06}^+ = 0, \quad l_{09-k}^+ = {}^1/_2\sigma_{11,k2}(\mathbf{v}^\circ; b_+, +0) \quad (k = 1, 2) \\ \mathbf{U}^7(\mathbf{y}) = [24\mu(\lambda+\mu)]^{-1}((\lambda+2\mu)y_1^3 - (3\lambda+4\mu)3y_1y_2^2, \\ 3\lambda y_1^2y_2 + (3\lambda+2\mu)y_2^3) \\ \mathbf{U}^8(\mathbf{y}) = [12\mu(\lambda+\mu)]^{-1}(-(3\lambda+4\mu)y_2^3 + (\lambda+2\mu)3y_1^2y_2, \\ 3\lambda y_1y_2^2 - (\lambda+2\mu)y_1^3) \end{gathered} \tag{7.1}$$

The right-hand sides of the boundary conditions in problem (2.2) are determined, for the vector function $\mathbf{v}^2$, by formulas (2.3) and $q_j^+(x_1) = {}^1/_2\sigma_{1j,12}(\mathbf{v}^\circ; x_1, +0)$, $x_1 \in [-b_-, b_+]$. Relations (2.6) and (2.9) in which the index 1 must be replaced by 2 hold, as well as the representation

$$\begin{gathered} \mathbf{v}^2(\mathbf{x}) = \sum_{i=1}^4 l_{2i}^+\mathbf{U}^i(\mathbf{y}) + l_{25}^+\mathbf{S}^{(1)}(\mathbf{y}) + l_{08}^+\boldsymbol{\Upsilon}^1(\mathbf{y}) - l_{07}^+\boldsymbol{\Upsilon}^3(\mathbf{y}) \\ (\Upsilon_r^3(r,\theta), \quad \Upsilon_\theta^3(r,\theta)) = -r[4\mu\pi]^{-1}\{\ln r[1+\varkappa](0,1) - \\ [\varkappa-1]\theta(1,0) + 2\varkappa[\varkappa-1]^{-1}(0,1) + (\sin 2\theta, \cos 2\theta)\} \end{gathered}$$

Relations (2.11) and (2.12) hold for the vector $\mathbf{v}^3$ (with the corresponding change of the indices from 1, 2 to 2, 3). We note that the relations $\sigma_{jj}(\mathbf{v}^\circ; x_1, +0) = 0$ yield the formulas $\overset{\circ}{v}_{1,1}(x_1, +0) = 0$ and $l_{01}^+ = l_{01}^-$.

The expansion of the solution of the problem in a thin strip has the form (4.1), and

$$\begin{gathered} w^{-1}(x_1) = 0, \quad w_1^\circ(x_1) = 0, \quad w_2^\circ(x_1) = t_3x_1^3 + t_2x_1^2 + t_1x_1 + t_0 \\ t_3 = B^{-2}(l_{03}^+ + l_{03}^-) - 2B^{-3}(l_{02}^+ - l_{02}^-), \quad t_2 = {}^1/_2B^{-1}(l_{03}^+ - \\ l_{03}^-) - {}^3/_2(b_+ - b_-)t_3 \end{gathered}$$

The coefficients of expansion in the boundary layer are

$$\begin{gathered} \mathbf{Z}^{0\pm}(\xi) = l_{01}^\pm\mathbf{e}^1 + l_{02}^\pm\mathbf{e}^2, \quad \mathbf{Z}^{1\pm}(\xi) = l_{03}^\pm\mathbf{U}^3(\xi), \quad \mathbf{Z}^{2\pm}(\xi) = l_{05}^\pm\mathbf{U}^5(\xi) + \\ l_{25}^\pm\mathbf{Z}^3(\xi) + l_{30}^\pm\mathbf{Z}^5(\xi), \quad \mathbf{Z}^{3\pm}(\xi) = l_{07}^\pm\mathbf{Z}^7(\xi) + l_{08}^\pm\mathbf{Z}^8(\xi) + l_{34}^\pm\mathbf{Z}^4(\xi) \end{gathered}$$

Here $\mathbf{Z}^7$ and $\mathbf{Z}^8$ are the solutions of the homogeneous problem (3.1) determined over the vector fields $\mathbf{U}^7$ and $\mathbf{U}^8$ (see (7.1)), just like the solution $\mathbf{Z}^6$ over $\mathbf{U}^6$.

As a result we obtain the following representation for SIF:

$$\begin{gathered} k_j^\pm(\varepsilon) = \varepsilon^{3/2}(l_{25}^\pm F_{3j}^\pm + l_{30}^\pm F_{5j}^\pm) + O(\varepsilon^{5/2}|\ln\varepsilon|) \\ l_{25}^\pm = \pm{}^1/_2\sigma_{11,2}(\mathbf{v}^\circ; \pm b_\pm, +0); \quad l_{30}^\pm = \mp{}^7/_{12}\sigma_{11,2}(\mathbf{v}^\circ; \pm b_\pm, +0) + \\ D(b_+ + b_-)^{-2}(6(v_2^\circ(+b_+, +0) - v_2^\circ(-b_-, +0)) - (b_+ + b_-)(\overset{\circ}{v}_{2,1}(\pm b_\pm, +0) - \\ \overset{\circ}{v}_{1,2}(\pm b_\pm, +0) + \overset{\circ}{v}_{2,1}(b_+, +0) - \overset{\circ}{v}_{1,2}(b_+, +0) + \overset{\circ}{v}_{2,1}(-b_-, +0) - \\ \overset{\circ}{v}_{1,2}(-b_-, +0)) \end{gathered}$$

Specification of the asymptotic equations leads to representation (6.8).

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EXTREMAL CRITERIA OF THE STABILITY OF CERTAIN MOTIONS*

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The so-called extremal criteria of the stability of certain types of motion were formulated in a number of publications /1, 3, 4, 7-10, 13, 16, 23-25, 31-34, 36-40, 44/. However, until now, the connection between these criteria has not been discussed, nor the problem of the possibility of extending them to embrace the wider classes of systems and motions considered. In a number of cases it might be found that the results of various investigations are contradictory.

In this connection the present paper combines a comparative survey of the work dealing with extremal criteria of stability, with a derivation (in cases when it was not already done) of the criteria in question in a unique manner, using the Poincare-Lyapunov small-parameter method. It should be noted that the same results can be obtained, under somewhat different assumptions, by the method of direct separation of motions. Three classes of systems are specified for which the extremal criteria of stability have been successfully established up to the present time. The basic results are given in the form of theorems. The applications of extremal criteria to the problem of deriving a general justification for the tendency for certain classes of weakly connected dynamic objects to synchronize, to the problems of designing new

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